NATURAL OSCILLATIONS IN COUETTE FLOWS OF AN INCOMPRESSIBLE MAXWELLIAN FLUID

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 103-105, 1966

One of the most interesting effects accompanying flows of polymer systems is the so-called "disintegration of the melt." When this phenomenon occurs, a jet of molten polymer or concentrated polymer solution ejected from a capillary suffers gross deformations of shape when the parameters determining the process of deformation reach certain critical values. Starting from the idea of specific elastic instability in flows of viscoelastic media, reference [1] gives criteria for the onset of this phenomenon and shows its applicability through analysis of a large amount of experimental data.

Slippage of a visoelastic fluid along the wall when moving at sufficiently high speeds is another possible cause of this effect. The problem of natural oscillations which appear in slippage of an incompressible Maxwellian fluid in Couette flows is introduced in this article in order to present a qualitative analysis of this mechanism of the appearance of irregularities in flows.

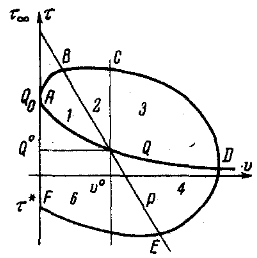

Fig. 1

Unfortunately, the literature lacks satisfactory data which would establish any quantitative laws of slippage along the wall during rapid flows of melts or solutions of polymers. However, we may assume that the laws describing this phenomenon are analogous or very close to those established for dry friction of viscoelastic materials. A typical curve of the dependence of the frictional force Q of such materials on the slippage velocity v has a maximum which shifts to the left when the normal pressure is increased [2, 3]. For the sake of simplicity, we shall limit ourselves here to considering the case in which the frictional force has a decreasing velocity characteristic on the assumption that $d^2Q/dv^2 > 0$, $Q(v) > 0$, $\lim Q = 0$ when $v \to \infty$. This sort of assumption corresponds to a very high hydrostatic pressure in the system.

In addition, we assume that irregularity in the process of friction which appears with time-dependent v is not important, that is, the frictional force can be described by a steady relationship Q(v).

We shall formulate the basic assumptions in regard to processes of slippage along the wall. We assume that the fluid is loosened from the walls when the tangential stress on it reaches a critical value $Q(0) = Q_0$. The quantity Q_0, which is a measure of the strength of adhesion of a viscoelastic fluid to the material of the wall, depends, in particular, on the characteristics of this fluid—density ρ, viscosity η, and relaxation time θ. Since the density of the majority of polymers is approximately the same, $Q_0 = Q_0(\eta, \theta)$. Dimensional analysis shows that $Q_0 \sim \eta/\theta = G$. Consequently, the characteristic Q(v) also undergoes some changes with changes in the material constants, that is, if we give a certain curve for Q(v), we simultaneously give the parameters ρ, η, and θ. Assuming, for the sake of simplicity, that the distance 2h between plates is also fixed, we obtain a single parameter which can be varied—the relative velocity of the plates 2V.

It is clear from symmetry considerations that the problem of flows of a Maxwellian fluid between parallel plates moving in opposite directions at speeds V and $-V$ can be solved in the interval $0 \le y \le h$, assuming that the speed is always zero on the lower boundary. The velocity field and the tangential stresses are determined from solving the system of equations

$$\rho \frac{\partial v_x}{\partial t} = \frac{\partial p_{xy}}{\partial y}, \qquad \theta \frac{\partial p_{xy}}{\partial t} + p_{xy} = \eta \frac{\partial v_x}{\partial y}. \tag{1}$$

The system of boundary and initial conditions is of the form

$$v_x|_{t=0} = 0, \quad p_{xy}|_{t=0} = 0, \qquad v_x|_{y=h} = 0, \quad v_y|_{y=h} = V - v(t). \tag{2}$$

When $v(t) \neq 0$, we have another relationship in addition to the last condition

$$p_{xy}|_{y=h} = Q(v) \qquad (v \neq 0). \tag{3}$$

The problem (1)-(3) is a linear problem with nonlinear boundary conditions. As we intend to obtain simple semiquantitative results, we shall solve it approximately here by averaging the quantities v_x and p_{xy} on the interval [0, h]. Making use of the results of reference [4], we can show that the velocity distribution is of the form

$$v_x(t, y) = U(t)\left[\frac{y}{h} + O\left(\frac{t_0}{\theta}\right)\right]. \tag{4}$$

Here U(t) is the variable velocity of the fluid at the movable plate and $t_0 = h(\rho\theta/\eta)^{1/2} \ll \theta$ is the time required for elastic waves to propagate from the plate $y = h$ to the plane $y = 0$. Averaging equation (1) and taking the boundary conditions into consideration, we obtain

$$\theta \frac{d\tau}{dt} = \tau_\infty - \tau, \qquad \tau_\infty = \frac{\eta}{h} V \qquad (v = 0), \tag{5}$$

$$\theta \frac{d\tau}{dt} = \frac{\eta}{h}(V - v) - \tau, \qquad \frac{\rho h}{2}\frac{dv}{dt} = \tau - Q(v) \quad (v \neq 0). \tag{6}$$

We shall seek the solution of equations in the class of continuous piecewise smooth functions in the right half-plane of the phase plane (v, τ) containing the axis $v = 0$.

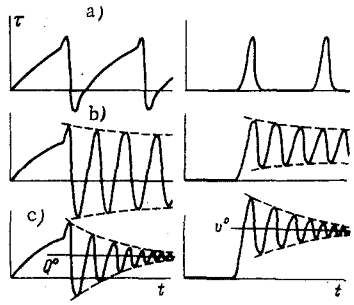

Fig. 2

On condition that $\tau(0) = 0$, we obtain from equation (5)

$$\tau(t) = \tau_\infty (1 - e^{-t/\theta}). \tag{7}$$

Taking account of the hypothesis concerning the nature of slippage along the wall which was formulated previously, it follows that slippage of the fluid along the wall begins only when

$$V > V_* = hQ_0/\eta.$$

Let us consider the case $V > V^*$. When $t = t_* = -\theta \ln(1 - \theta h/\eta V_*)$, the stress on the wall reaches the critical magnitude Q_0 and motion with slippage begins which is described by equations (6). The initial conditions for this system is of the form

$$v|_{t=t_*} = 0, \qquad \tau|_{t=t_*} = \tau_* = Q_0.$$

The phase trajectory of the process can also be obtained from an equation following from (6)

$$\frac{d\tau}{dv} = \frac{\rho h}{2\theta}\frac{P(v) - \tau}{\tau - Q(v)}, \qquad P(v) = \frac{\eta}{h}(V - v) \tag{8}$$

with the initial condition $\tau(0) = Q_0$.

Equation (8) has a singularity (Q°, v°) determined, as is readily seen, by the intersection of the curves Q(v) and P(v). The properties of Q(v) cited previously imply that when V increases, v° tends toward V, but remains smaller than V, and the quantity $Q^\circ = Q(v^\circ)$ tends toward zero (refer to Fig. 1).

The lines $\tau = P(v)$ constitute a one-parameter family of straight lines with different slopes. Considering the linearized equations (6) in the neighborhood of the singularity (Q°, v°) in the usual manner and taking account of the properties of Q(v), we find that depending on the magnitude of V, four types of singularities are theoretically possible: a) a stable focus when $Q'^\circ = |dQ/dv|(v = v^\circ) < \rho h/2\theta$; b) an unstable focus when $\rho h/2\theta < Q'^\circ < 2\rho\eta/\theta)^{1/2}$; c) a center when $Q'^\circ = \rho h/2\theta$; and d) an unstable node when $Q^{10} \geq (2\rho\eta/\theta)^{1/2}$ (in case of equality, a degenerate node is produced).

By making use of (8), it is also easy to determine the qualitative behavior of the integral curve from point A$(0, Q_0)$. In the curvilinear sector 1 bounded by the straight line P(v) and the curve Q(v), the function $\tau(v)$ increases from Q_0 at point A to max[τ] at point B (Fig. 1), having an unbounded derivative at point A. Function $\tau(v)$ decreases very slowly ($\tau'(v) \approx -\rho h/2\theta$) in sector 2 on section BC. In sector 3, $\tau(v)$ decreases more rapidly down to the intersection of the curve $\tau(v)$ with the characteristic Q(v) at point D, where the maximum slippage velocity max [v] is reached. Further, in sector 4 there is an additional decrease in $\tau(v)$ to the value of min [τ] which occurs at point E of the phase trajectory. In sector 5, the values of $\tau(v)$ increase slightly ($\tau'(v) \approx \rho h/2\theta$); this increase is accelerated in sector 6. Two cases are possible—either $\tau(v)$ intersects the characteristic Q(v) when $0 < v < \max[v]$, or $\tau(v)$ intersects the axis v = 0 at some point $\tau = \tau^*$ (the second case is shown in Fig. 1). In the first case, the further behavior of the phase trajectory is wholly determined by the type of singularity ($v^{*\circ}$, $Q^{*\circ}$), that is, either a stationary mode of flow with slippage is established if this point is a stable focus, or the phase trajectory is rotated a limit cycle and a self-oscillating mode without adhesion appears. In the second case, if the phase trajectory reaches the axis v = 0 at point τ^*, the process is either again described by equation (5) if $|\tau^*| \approx Q_0$, that is, the representative point moves upward along the axis v = 0 to separation at the point Q_0 in the right half-plane, or, if $|\tau^*| \approx Q_0$, the trajectory goes into the region $v < 0$. If the phase trajectory reaches the axis v = 0 at the time $t = t^*$ at the point $|\tau^*| \leq Q_0$, then, when $t > t^*$, we have

$$\tau(t) = \tau^* \exp\left(-\frac{t - t^*}{\theta}\right) + \tau_\infty\left[1 - \exp\left(-\frac{t - t^*}{\theta}\right)\right], \tag{9}$$

$$v(t) = 0 \qquad (t > t^*). \tag{9 cont'd}$$

When the function $\tau(t)$ reaches the value Q_0, separation occurs again and the cycle described here is repeated. The self-oscillations appearing in this case are of a relaxation type (Figs. 1, 2a).

If $\tau^* < -Q_0$, motions with negative slippage velocity are possible, and the self-oscillations appearing in this instance are similar to those appearing in solid dry friction [5].

Detailed evaluations based on investigations of majorizing equations show that the following sequence of changes in modes of motion when V exceeds the value V^* is possible: 1) stationary flow with adhesion ($V < V^*$); 2) relaxation self-oscillations with adhesion; 3) self-oscillations with a shift to the left half-plane of the phase plane; 4) relaxation self-oscillations reappear; and 5) steady motion with slippage. We note that the foregoing are associated with cases in which the characteristic Q(v) has the above-mentioned properties; the series of modes noted above may not appear in some forms of Q(v). On the other hand, if the absolute value of the derivative dQ/dv remains sufficiently high, even for large v, a self-oscillating mode without adhesion may appear between the fourth and fifth modes. Diagrams of changes in the slippage velocity and the stresses in different modes are presented in Fig. 2; relaxation self-oscillations are shown in Fig. 2a, self-oscillations without adhesion in Fig. 2b (the phase trajectory is included in the open region $v > 0$ of the phase plane), and the establishment of a stationary mode of slippage in Fig. 2c.

In conclusion, the authors thank G. I. Barenblatt for discussions and his constant interest in this work.

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27 December 1965 Moscow